ECE454/544: Fault-Tolerant Computing & Reliability Engineering

Lecture #7 –

Probability & Random Variables (Review)

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Administrative Issues

- Homework#2 solution posted
- Homework#3
  - Due **Oct. 5, Wednesday**
- Project Proposal
  - Due **Oct. 5, Wednesday**
  - Refer to Proposal Guideline on the course website

Fault-Tolerant Computing (Review)

- How to design fault-tolerant systems?
  - Redundancy: the addition of information, resources or time beyond what is needed for normal system operation, to detect and possibly tolerate fault
    - Hardware redundancy
    - Information redundancy
    - Time redundancy
    - Software redundancy

Lectures #2 ~ 6

Next…

- Reliability modeling and evaluation
  - To evaluate the inherent reliability of a product or process and pinpoint potential areas for reliability improvement.
  - To identify the most likely failures and then identify appropriate actions to mitigate the effects of those failures → sensitivity analysis.

Quantitative Reliability Measures

- Failure function *F(t)*
- Reliability function *R(t)*
- Failure rate *z(t)*
- Mean time to failure (MTTF)
- Mean residual life (MRL)

Time-to-Failure (T)

- A *random variable* describing the time elapsing from when a component is put into operation until it fails for the first time

Review of Random Variables & Related

Random Experiment ε

- Outcome is *unknown* in advance
- Set of all possible individual outcomes is *known*

- Example: tossing a die

Sample Space Ω

- The set of all possible individual outcomes (sample points / elementary events) of an experiment

- Different types:
  - Finite *vs*. infinite
  - Discrete *vs*. continuous
    - Discrete: sample points can be put into 1-to-1 correspondence with positive integers
    - Continuous: sample points consist of all numbers on some finite or infinite interval of real line

- **Example:** sample space for "tossing a die"? Type?

Events E

- A subset of a sample space
- Example:
  - $E1$={outcome is prime}={2, 3, 5}: the event of rolling a prime number

- E may be Ω or $\varnothing$ (impossible event)
- We say an event E occurs if the random experiment is performed and the observed outcome ω is in E, i.e., $\omega \in \boldsymbol{E}$.

"Axioms of Probability"

- For each event E of sample space Ω, if a number $P(E)$ is defined and satisfies the following axioms

  A1: $0 \le P(E) \le 1$

  A2: $P(\Omega) = 1$

  A3: for any sequence of pair-wise-mutually-exclusive events, $E_1, E_2, \ldots$ (i.e. $E_i E_j = \varnothing$ for any $i \neq j$), we have

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

$P(E)$ is refer to as the **probability** of the event E

Conditional Probability

- Definition: Let A and B be two events, then the **conditional probability** of A given B: P[A|B] is a number such that

  1. $0 \le P[A \mid B] \le 1$
  2. $P[A \cap B] = P[B] P[A \mid B]$

  Note:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

- **Exercise:** "Tossing a die" - find the conditional probability of the event of the uppermost side showing 3 spots given the event of rolling an odd number occurring?

Law of Total Probability

- Let $\{B_i\}_{i=1}^n$ be a *partition* of Ω, then

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

$$P(A) = P(A \mid B) P(B) + P(A \mid \overline{B}) P(\overline{B})$$

Bayes' Theorem

- Based on Total Probability Theorem, we have "Bayes' Theorem / Rule / Formula"
  - Let $\{B_i\}_{i=1}^n$ be a *partition* of Ω, then for any event A with P(A)>0, we have

$$P(B_i \mid A) = \frac{P(A \mid B_i) P(B_i)}{P(A)} = \frac{P(A \mid B_i) P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j) P(B_j)}$$

Hands-on Problem

- An on-line computer system has five incoming communication lines with the properties described in the following table.

Line	Fraction of traffic	Fraction of message without error
1	0.1	0.97
2	0.1	0.96
3	0.4	0.99
4	0.1	0.97
5	0.3	0.98

  - What is the probability that a randomly chosen message has been received without error?
  - Suppose a message selected at random is found to be free of error. What is the probability that this message was from communication line 3?

Independence

- **Definition**: Events A and B are said to be independent if

$$P(A \cap B) = P(A) \bullet P(B)$$

- Neither event influences the occurrence of the other: $P(A|B)=P(A)$, $P(B|A)=P(B)$
- **Being independent <> Being mutually exclusive**

Independence of a set of events

- A list of n events $A_1, A_2, \ldots, A_n$ is defined to be "**mutually independent**" *iff* for each set of k ($2 \le k \le n$) distinct indices $i1, i2, \ldots, ik$, which are elements of $\{1,2,\ldots,n\}$, we have

$$P(A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}) = P(A_{i1})P(A_{i2})\ldots P(A_{ik})$$

- If events $A_1, A_2, \ldots, A_n$ are such that every pair is independent, then such events are called "**pair-wise independent**"

Independence of a set of events (Cont'd)

- **Mutually independent <> pair-wise independent**

- A set of events is pair-wise independent. It does not follow that the list of events is mutually independent!

Hands-on Problem

Consider the experiment of rolling two dice. Let the sample space $S=\{(i, j) \mid 1<=i, j <=6\}$. Also, assume that each sample point is assigned a probability of 1/36. Define the events A, B, and C so that

A = "first die results in a 1, 2, or 3"

B = "second die results in a 4, 5, or 6"

C = "the sum of the two faces is 7"

Are events A, B, and C mutually independent?

Are they pair-wise independent? Justify your answer.

Random Variable

- Informally, a random variable (*r.v.*) X is a real-valued function from some sample space Ω to R, i.e., $X: \Omega \rightarrow R$
- A *r.v.* X maps each outcome ω in Ω *to a* real number $X(\omega) \in R$
- *r.v.* is not *random* or *variable*, but a *function*
- The mapping is not random but deterministic

Random Variable Example

- *"tossing a fair coin three time"*
  - Ω = {TTT; TTH; THT; THH; HTT; HTH; HHT; HHH}
  - Let X be the number of heads tossed in 3 times
  - We can map each outcome in Ω to a real number:

Distribution Function

- Definition: The cumulative distribution function (c.d.f.) or more simply the distribution function F of a *r.v.* X is defined for each real number x, by

$$F(x) = P[\{\omega : \omega \in \Omega \text{ AND } X(\omega) \le x\}] \\ = P\{X \le x\}$$

- Property:
  - F is a non-decreasing function

    If $x < y$ then F(x) <= F(y)
  - For all $x < y$, $P\{x < X \le y\} = F(y) - F(x)$
  - Also,

$$\lim_{x \to \infty} F(x) = 1 \qquad \lim_{x \to -\infty} F(x) = 0$$

Discrete vs. Continuous *r.v.*

- A r.v. that can take on (at most) a countable number of possible values is said to be a ***discrete*** r.v.

  Example: tossing coin example

- A r.v. that can take on a range of real values (uncountable!) is said to be a ***continuous*** r.v.

  Example: the time of the failure of a component (in general, the time when a particular event occurs)

  - Probability density function (p.d.f.)
  - Parameters: Expectation/Mean: E[X], the *k*th moment, Variance

Continuous *r.v.s* (1)

- Probability density function (p.d.f.)

  The p.d.f. denoted by $f_X(x)$ for a continuous *r.v.* X is defined by

$$f_X(x) = F'_X(x) = \frac{dF_X(x)}{dx}$$

- Property:

$$F_X(a) = P(X \le a) = \int_{-\infty}^{a} f_X(x)dx$$

$$F_X(\infty) = \int_{-\infty}^{\infty} f_X(x)dx = 1$$

$$\int_a^b f_X(x)dx = P(a \le X \le b) = F_X(b) - F_X(a)$$

Continuous *r.v.s* (2)

- Expectation/Mean/Expected value of X

  Let X be a continuous *r.v.* with p.d.f. $f_X(x)$, the mean of X, denoted by μ=E[X], is defined by

$$\mu = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Expectation of a function of a r.v. X

  For any real-valued function *g:*

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

- The *k*th moment of X

  The *k*th moment of X is defined by $E[X^k]$, *{k = 1, 2, 3, …}.*

$$E[X^k] = \int_{-\infty}^{\infty} x^k f_X(x) dx$$

Continuous *r.v.s* (3)

- Variance of X

  A measure of its statistical dispersion, indicating how far from the expected value its values typically are (Wikipedia 2006)

  If X is a r.v. with mean μ, then the variance of X, denoted by Var(X)= δ^2, is defined by **Var(X)= E[(X-μ)²]**

  – If X is a continuous *r.v.*, then

$$Var[X] = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) dx$$

  – An alternative formula:

$$Var[X] = E[X^2] - (E[X])^2 = E[X^2] - \mu^2$$

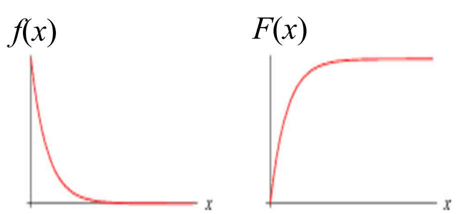
An Example Continuous *r.v.*

- Exponential r.v.: a continuous r.v. X has an exponential distribution with parameter λ if its *p.d.f*:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

  - *C.d.f:*

$$F(x) = \int_{-\infty}^{x} f(x)dx = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



  - E[X]=1/λ
  - Var[X]=1/λ^2

Next Topic

- Time-to-Failure Models and Distributions

Things to do

- Homework
- ECE544 Project Proposal
  - Due **Wednesday, Oct. 5**