ECE454/544: Fault-Tolerant Computing & Reliability Engineering

Lecture #8 –

Time-to-Failure Models and Distributions

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Fall 2022

Administrative Issues

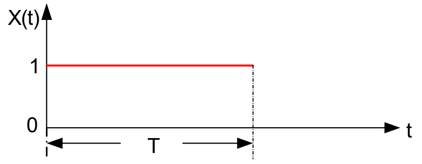
- Homework#3 assigned and due **Oct. 5, Wednesday**
- ECE544 Project Proposal
  - Due **Oct. 5, Wednesday**
  - Refer to Proposal Guideline on the course website

Learning Objectives

- Describe the concept of time-to-failure, failure function, reliability/survivor function, failure/hazard rate, MTTF, and MRL
- Understand their relationship
- Understand the role of exponential distribution in reliability studies

Time-to-Failure (T)

- A random variable describing the time elapsing from when a component is put into operation until it fails for the first time
- The state variable X(t)
  - 1: if the component is functioning at time t
  - 0: if the component fails at time t
- Relationship between T and X(t) of a component:



Quantitative Reliability Measures

- Failure function $F(t)$
- Reliability function $R(t)$
- Failure rate $z(t)$
- Mean time to failure (MTTF)
- Mean residual life (MRL)

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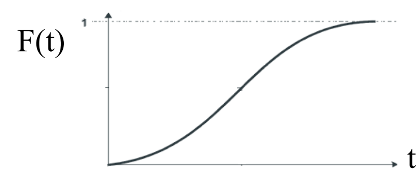
Failure Function F(t)

- The cumulative distribution function (c.d.f) of the *r.v.* T

$$F_T(t) = P\{T \le t\}$$

  - $0 <= F(t) <= 1$
  - $F(t)$ is a non-decreasing function

    If $t1 < t2$ then $F(t1) <= F(t2)$



- Probability that the unit fails within the time interval (0, t] (assuming t=0 is the starting point)

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P.D.F. of T: *f(t)*

- Probability density function (P.D.F.)

$$f_T(t) = F_T^{'}(t) = \frac{dF_T(t)}{dt}$$
$$= \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{P\{t < T \le t + \Delta t\}}{\Delta t}$$

  – When Δt is very small:

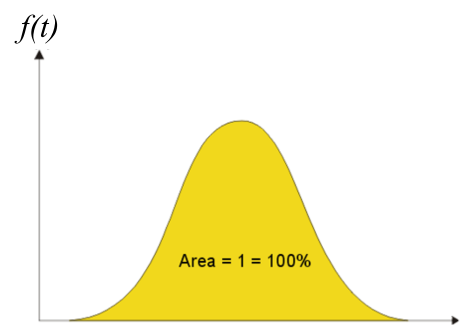
$$P\{t < T \le t + \Delta t\} \approx f_T(t) \bullet \Delta t$$

F(t) & f(t)

- For $t >= 0$:

$$F_T(t) = P\{T \le t\} = \int_0^t f_T(x)dx$$

$$F_T(\infty) = \int_0^\infty f_T(x)dx = 1$$

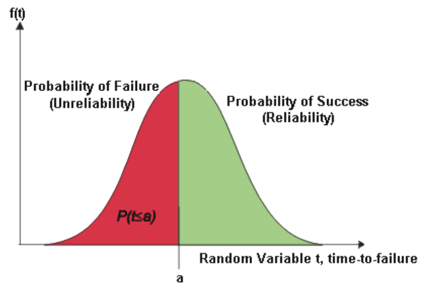


Reliability Function R(t)

- Review: the conditional probability that a system performs correctly throughout an interval of time $[t_0, t]$, given that it was performing correctly at time t_0
- Reliability / survivor function: for t>0

$$R(t) = 1 - F(t) = P\{T > t\}$$

  - Probability that the unit does not fail within the time interval (0, t]
  - Probability that the unit survives [0, t] and is still functioning at t.



Failure Rate *z(t)*

- A measure of the instantaneous speed of failure: # of failures per time unit
- Also, hazard rate or hazard function

$$z(t) = \lim_{\Delta t \to 0} \frac{P\{t < T \le t + \Delta t \mid T > t\}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{R(t)\Delta t} = \frac{f(t)}{R(t)}$$

  - When Δt is very small:

$$P\{t < T \le t + \Delta t \mid T > t\} \approx z(t) \bullet \Delta t$$

z(t) vs. *f(t)*

$$P\{t < T \le t + \Delta t\} \approx f_T(t) \bullet \Delta t$$

$$P\{t < T \le t + \Delta t \,|\, T > t\} \approx z(t) \bullet \Delta t$$

- Suppose we start out a new item at t=0 and at t=0 ask:
- "What is the prob. that this item will fail in the interval (t, t+ Δt]?"

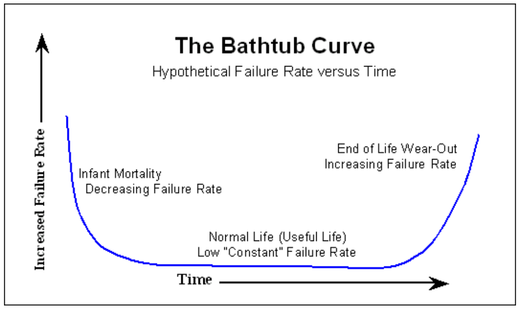
$$P\{t < T \le t + \Delta t\} \approx f_T(t) \bullet \Delta t$$

- The item has survived until time t and ask:"What is the prob. that this item will fail in the next interval (t, t+ Δt]?"

$$P\{t < T \le t + \Delta t \,|\, T > t\} \approx z(t) \bullet \Delta t$$

z(t): The Bathtub Curve

- The lifetime of an item may be divided into 3 phases:
  - Burn-in/early-life/infant mortality period
  - Useful life period
  - Wear-out period



www.weibull.com

Relationship between F(t), f(t), R(t) and z(t)

Expressed by	F(t)	f(t)	R(t)	z(t)
F(t)=	X			
f(t)=		X		
R(t)=			X	
z(t)=				X

$$F(t) = \int_0^t f(x)dx = 1 - R(t) = 1 - e^{-\int_0^t z(u)du}$$

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} = z(t)e^{-\int_0^t z(u)du}$$

$$R(t) = 1 - F(t) = \int_t^\infty f(x)dx = e^{-\int_0^t z(u)du}$$

$$z(t) = \frac{dF(t)/dt}{1 - F(t)} = \frac{f(t)}{\int_t^\infty f(x)dx} = -\frac{d}{dt}\ln R(t)$$

Mean Time To Failure (MTTF)

- The expected time that a component/system will operate before the first failure

$$MTTF = E[T] = \int_0^\infty tf(t)dt$$

$$MTTF = \int_0^\infty R(t)dt$$

- MTTR: mean time to repair a component
- MTBF: mean time between failures

MTBF = MTTF+MTTR

Mean Residual Life (MRL)

- Conditional survivor function at age t:

$$R(x \mid t) = \Pr(T > x + t \mid T > t) = \frac{R(x+t)}{R(t)}$$

- Mean residual life at age t:

$$MRL(t) = \int_0^\infty R(x \mid t)dx = \frac{1}{R(t)} \int_t^\infty R(x)dx$$

$$MRL(0) = MTTF$$

Learning Objectives (revisit)

✓ Describe the concept of time-to-failure, failure function, reliability/survivor function, failure/hazard rate and MTTF

✓ Understand their relationship

- Understand the role of exponential distribution in reliability studies

Time-to-Failure Distribution

- In practice, T can have any arbitrary distributions
  - Exponential ⬅
  - Weibull
  - Normal
  - Lognormal
  - Gamma
  - etc

Exponential Distribution (1)

- The most commonly used distribution in applied reliability analysis
  - The mathematical simplicity
  - Leading to realistic lifetime models
- **Definition:** assume T is exponentially distributed with parameter λ >0. The pdf of T is:
  - $$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0, \lambda > 0 \\ 0, & t < 0 \end{cases}$$

Exponential Distribution (2)

- Failure function:

$$F(t)=\begin{cases}1-e^{-\lambda t}, & t\ge 0\\ 0, & t<0\end{cases}$$

- Reliability function:

$$R(t)=e^{-\lambda t}\quad \text{for}\quad t\ge 0$$

- Failure rate:

$$z(t)=\frac{f(t)}{R(t)}=\lambda$$ **Constant!**

- MTTF:

$$MTTF=\int_0^{\infty}R(t)dt=\frac{1}{\lambda}$$

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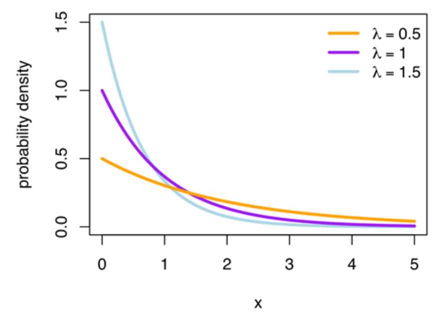
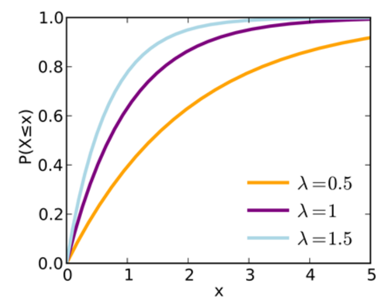
Exponential *pdf* and *cdf*





Source: https://en.wikipedia.org/wiki/Exponential_distribution

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Exponential Distribution (3)

- Memoryless/Markov property
  - The future is independent of the past!
  - The remaining lifetime of a component, which functions at time *t*, is independent of *t*

$$R(h \mid t) = P\{T > t + h \mid T > t\}$$
$$= P\{T > h\} = \mathrm{R(h)} \quad \forall t, h > 0$$

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Exponential Distribution (4)

- Memoryless/Markov property
  - A used item is stochastically as good as new as long as it is still functioning

$$MRL(t) = \int_0^\infty R(x \mid t)dx = \int_0^\infty R(x)dx = MTTF$$

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Hands-On Problem (1)

- A rotary pump has a constant failure rate $\lambda=10^{-4}$/hour [Rausand03]
  - What is the probability that the pump survives 1 month in continuous operation?
  - What is the MTTF of the pump?
  - Suppose the pump has been functioning without failure during its first 2 months in operation. What is the probability that the pump will fail during the next month?

Hands-on Problem (2)
Mixture of Exponential Distributions

- Assume that the same type of items are produced at two different plants. The items are independent and have constant failure rates. The production process is slightly different at two plants and items will have different failure rates (λ_i: for items from plant i). The items are mixed up before being sold. A fraction p is coming from plant 1, and the rest is from plant 2.
  - If we pick one item at random, what is the reliability function of this item?
  - What is the MTTF of the item?

Hands-on Problem (3)

Non-constant Failure Rate

- A component with time to failure T has failure rate $z(t) = 2.0*10^{-6}\, t$/hour for $t>0$
  - Determine the probability that the component survives 200 hours
  - Determine the probability that a component, which is functioning after 200 hours, is still functioning after 400 hours

Relationship between F(t), f(t), R(t) and z(t)

Expressed by	F(t)	f(t)	R(t)	z(t)
F(t)=	X			
f(t)=		X		
R(t)=			X	
z(t)=				X

$$F(t) = \int_0^t f(x)dx = 1 - R(t) = 1 - e^{-\int_0^t z(u)du}$$

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} = z(t)e^{-\int_0^t z(u)du}$$

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$$z(t) = \frac{dF(t)/dt}{1 - F(t)} = \frac{f(t)}{\int_t^\infty f(x)dx} = -\frac{d}{dt}\ln R(t)$$

Summary of Lecture#8

- Quantitative measures
  - Time to failure (T): a random variable describing the time elapsing from when a component is put into operation until it fails for the first time
  - Failure function F(t): the cumulative distribution function (c.d.f) of the r.v. T
  - Reliability/survivor function R(t)=1-F(t)
  - Failure rate (hazard rate/function) z(t)
  - Relationship between F(t), f(t), z(t), and R(t)
  - The bathtub curve for the failure rate
    - Burn-in/infant mortality period
    - Useful-life period
    - Wear-out period
  - Mean time to failure (MTTF)
  - Mean residual life (MRL)
- Exponential time to failure distribution has constant failure rate and memory-less property

Next Topic

- Fault trees analysis

Things to do

- Homework
- Project Proposal
  - Due **Wednesday, Oct. 5**