ECE454/544: Fault-Tolerant Computing & Reliability Engineering

Lecture #15 –

Reliability Analysis Using Markov Models (I)

Instructor: Dr. Liudong Xing

Fall 2022

Administrative Issues

- Homework#6
  - Due by **Nov. 7, Monday (Today)**
- **No ECE544 on Nov. 9, Wednesday**
  - Follow Friday's Class Schedule
  - Friday: Veterans Day Holiday; No Classes
- Project final report
  - Due by **Nov. 30, Wednesday**
  - Please check out the Report Guidelines for requirements

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Review of Lecture #14

- Special dynamic gates capture sequential dependencies arising in modeling fault tolerant systems
  - FDEP for modeling situations where one component's correct operation is dependent upon the correct operation of some other component
  - CSP for modeling cold spares which are unpowered before being used
  - WSP for modeling warm spares which fail at a reduced rate before being used
  - HSP for modeling hot spares which fail at active failure rate before being switched into active use
  - PAND for modeling ordered ANDing events
  - Two examples: HECS and FTPP

Reliability Models

- Fault trees (static)
- Reliability Block Diagrams (RBD)
- SDP, I/E based on minimal cut-sets
- SDP, I/E based on minimal path-sets
- Binary Decision Diagrams (BDD)

Analyze static systems whose failure can be expressed as the combination of component failures

— Combinatorial Models

Markov models

- Can evaluate dynamic fault trees, which are used for modeling sequence dependencies, priorities, cold/warm/hot spares etc.

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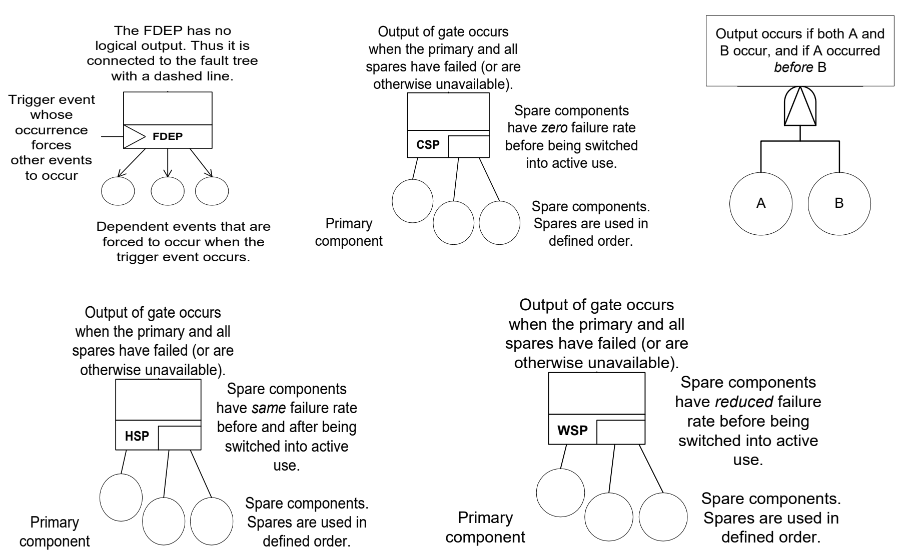
A Review of Dynamic Gates (L#15)



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Fault Trees and Markov chains

- Any static fault tree model (with exponential distributions) can be solved as a Markov chain.
  - In general, Markov chain solution is more time-consuming than the BDD (and in fact most other combinatorial approaches)
- Any dynamic fault tree model (with exponential distributions) can be solved as a Markov chain

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Agenda

- **Basic Concepts on Markov model**
- Reliability Analysis using Markov model

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Random Variable (Review, L#7)

- A random variable (*r.v.*) X is a real-valued function from some sample space Ω to R, *i.e., X:* $\Omega \rightarrow R$
- A *r.v.* X maps each outcome ω in Ω to a real number $X(\omega) \in R$

- Example: *"tossing a fair coin three time"*
  - Ω = {TTT; TTH; THT; THH; HTT; HTH; HHT; HHH}
  - Let X be the number of heads tossed in 3 times
  - We can map each outcome in Ω to a real number:

    X(TTT)=0; X(TTH)=1; X(THT)=1; X(THH)=2;

    X(HTT)=1; X(HTH)=2; X(HHT)=2; X(HHH)=3

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Stochastic Processes

- Definition:

  A family of r.v.'s {X(t) | t ∈T} that is indexed by a parameter *t* (such as time) is known as a "stochastic process" (or chance/random process)
  - Index set: T → the set of all possible values of t
    - Each element of T is referred to as a *parameter*
  - State space → the set of all possible values assumed by r.v.'s X(t)
    - Each of these values assumed by r.v. X(t) is called a *state* of the SP

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Markov Processes

- A Markov process is a stochastic process satisfying the *Markov property:* probabilities of future states depend only on the current state and not on how the process reached that state

$$P\{X(t+v)=j\,|\,X(t)=i;X(u)=x(u);0\le u<t\}$$
$$=P\{X(t+v)=j\,|\,X(t)=i\}$$
$$\forall x(u);0\le u<t$$

  – $\{X(t)=j\}$: event that the system is in state j at time t
  – $P_j(t)=P\{X(t)=j\}$: the probability of the event

- A discrete state Markov process is called a Markov chain

Markov Processes (Cont'd)

- State transition probability:

$$P\{X(t+v)=j\,|\,X(t)=i\}$$

- Stationary (steady-state) transition probability: if the transition probability does not depend on the time t but only on the time interval v for the transition

$$P\{X(t+v)=j\,|\,X(t)=i\}=P_{ij}(v), \quad \text{for } t,v>0$$

Be homogeneous in time!

Markov Processes (Cont'd)

- State transition probabilities satisfy:

$$\text{P1:} \quad 0 \le P_{ij}(t) \le 1, \quad \text{for } t > 0$$

$$\text{P2:} \quad \sum_{j=0}^{r} P_{ij}(t) = 1, \quad \text{for } i = 0, 1, 2, ..., \text{r}, \ \ \text{t} > 0$$

$$\text{P3:} \quad P_{ij}(t+v) = \sum_{k=0}^{r} P_{ik}(t) P_{kj}(v), \quad \text{for } t, v > 0$$

-- **Chapman-Kolmogorov equation, following from the Markov property & Total Probability Theorem**

Markov Processes (Cont'd)

- State transition rate from state *i* to state *j* is defined as:

$$\alpha_{ij} = \lim_{\Delta t \to 0} \frac{P\{X(t+\Delta t) = j \mid X(t) = i\}}{\Delta t} = \lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t)}{\Delta t}$$

Since $\dot{P}_{ij}(t) = \frac{dP_{ij}(t)}{dt} = \lim_{\Delta t \to 0} \frac{P_{ij}(t+\Delta t) - P_{ij}(t)}{\Delta t}$,

we have $\alpha_{ij} = \lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t)}{\Delta t} = \dot{P}_{ij}(0)$.

- Constant transition rate α_{ij} → the time the system is staying in state *i* until transition to state *j* is exponentially distributed with parameter α_{ij} !

Agenda

✓ Basic Concepts on Markov model
- **Reliability Analysis using Markov model**

Reliability Analysis Using Markov Chains

- Step 1: convert the fault tree model to a Markov chain
- Step 2: find the state equations of the Markov chain
- Step 3: find state probabilities by solving the state equations
- Step 4: find the system reliability or unreliability

Step 1: Fault Tree to Markov Chain Conversion

- Start with initial state in which all components are operational.
- For each operational state, enumerate all child states by considering effects of one component (call it *j*) failure at a time
- Establish transition rate from parent to child state as failure rate of component *j* (times the number of active replicates).
- Determine if "new" state is operational or failed.
- Continue until all operational states are enumerated.

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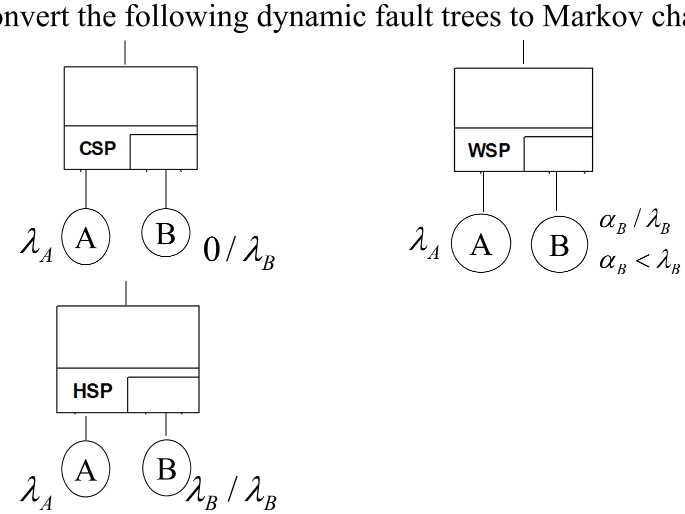
Hands-On Problems (1)

- Convert the following dynamic fault trees to Markov chains



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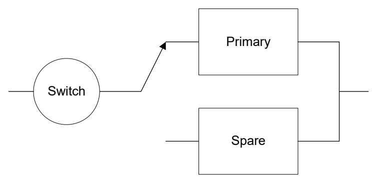
Hands-On Problems (2)

- Consider a two-component cold standby sparing system as shown in the follow figure:



- Find
  - Dynamic fault tree of the system, and
  - The Markov chain model

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Hands-On Problems (3)

- Consider a repairable system composed of a single component, the failure rate of the component is λ, the repair rate is μ. Find the Markov chain model.

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Reliability Analysis Using Markov Chains

- ✓ Step 1: convert the fault tree model to a Markov chain
- **Step 2: find the state equations of the Markov chain**
- Step 3: find state probabilities by solving the state equations
- Step 4: find the system reliability or unreliability

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State Equations of Markov Processes

$$\begin{bmatrix} -\alpha_{00} & \alpha_{10} & \alpha_{20} & \ldots \alpha_{r0} \\ \alpha_{01} & -\alpha_{11} & \alpha_{21} & \ldots \alpha_{r1} \\ \alpha_{02} & \alpha_{12} & -\alpha_{22} & \ldots \alpha_{r2} \\ : & : & : & : \\ \alpha_{0r} & \alpha_{1r} & \alpha_{2r} & \ldots -\alpha_{rr} \end{bmatrix} \bullet \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ : \\ P_r(t) \end{bmatrix} = \begin{bmatrix} \dot{P}_0(t) \\ \dot{P}_1(t) \\ \dot{P}_2(t) \\ : \\ \dot{P}_r(t) \end{bmatrix}$$

Or $\boldsymbol{A} \bullet \boldsymbol{P}(t) = \dot{\boldsymbol{P}}(t)$

Where ***A*** is the called the *transition rate matrix*:

$$A = \begin{bmatrix} -\alpha_{00} & \alpha_{10} & \alpha_{20} & \ldots \alpha_{r0} \\ \alpha_{01} & -\alpha_{11} & \alpha_{21} & \ldots \alpha_{r1} \\ \alpha_{02} & \alpha_{12} & -\alpha_{22} & \ldots \alpha_{r2} \\ : & : & : & : \\ \alpha_{0r} & \alpha_{1r} & \alpha_{2r} & \ldots -\alpha_{rr} \end{bmatrix}$$

For diagonal elements:

$$\alpha_{jj} = \sum_{k=0, k \neq j}^{r} \alpha_{jk}.$$

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State Equations – Derivation & Observations

- Observations
  - α_{jk}, k=0,1,…,j-1,j+1,…,r are transition rates from state j to the other states, called departure rates from state j. Then,
  - α_{jj} is the sum of the departure rates from state j
  - The sum of each column of $\boldsymbol{A}$ is ZERO
  - When the process enters state j, the system will stay in this state a time T_j, which is exponentially distributed with parameter α_{jj}. Thus the mean staying time in state j is

$$E(T_j) = \frac{1}{\alpha_{jj}} \quad \text{for} \quad j = 0,1,\ldots,r$$

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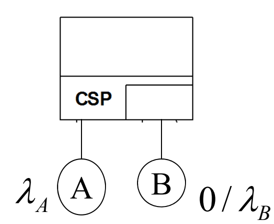
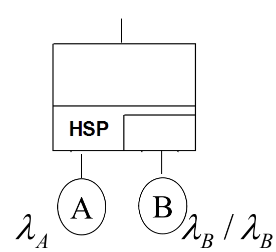
Hands-On Problems (4)

- Find the state equations of Markov chains for the
  - CSP system
  - HSP system
  - repairable system composed of a single component, the failure rate of the component is λ, the repair rate is μ.





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Reliability Analysis Using Markov Chains

- ✓ Step 1: convert the fault tree model to a Markov chain
- ✓ Step 2: find the state equations of the Markov chain
- **Step 3: find state probabilities by solving the state equations**
- Step 4: find the system reliability or unreliability

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Solving State Equations

- Unique solution to all the state probabilities $P_j(t)$ is obtained by solving

$$\begin{cases} \boldsymbol{A} \bullet \boldsymbol{P}(t) = \dot{\boldsymbol{P}}(t) \\ \sum_{j=0}^{r} P_j(t) = 1, \quad \text{the system must be in one of (r + 1) states} \\ P_i(0) = 1,\ P_k(0) = 0,\ \text{for } k = 0, \ldots, r \text{ and } k \neq i \quad \text{initial condition} \end{cases}$$

- Solutions
  - Asymptotic (steady-state/long-run) solution
  - Time-dependent solution

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Asymptotic Solution (1)

- The long-run (steady-state) probabilities, that is, $P_j(t)$ as t → ∞ are of interest in many applications

$$\lim_{t\to\infty} P_j(t) = P_j(\infty) = P_j,\ j = 0,1,\ldots,r$$

- If $P_j(t)$ → P_j (a constant), as t → ∞, then

$$\lim_{t\to\infty} \dot{P}_j(t) = 0,\ j = 0,1,\ldots,r$$

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Asymptotic Solution (2)

- The steady-state equations are therefore:

$$\begin{bmatrix} -\alpha_{00} & \alpha_{10} & \alpha_{20} & \ldots\alpha_{r0} \\ \alpha_{01} & -\alpha_{11} & \alpha_{21} & \ldots\alpha_{r1} \\ \alpha_{02} & \alpha_{12} & -\alpha_{22} & \ldots\alpha_{r2} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{0r} & \alpha_{1r} & \alpha_{2r} & \ldots-\alpha_{rr} \end{bmatrix} \bullet \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Or $A \bullet P = 0$

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Asymptotic Solution (3)

- To calculate the steady-state probabilities $P_{j,}$ use r of (r+1) linear algebraic equations and the fact that the sum of state probabilities always is equal to 1, that is,

$$\sum_{j=0}^{r} P_j(t) = 1$$

- The initial state of the process has no influence on the steady-state probabilities!

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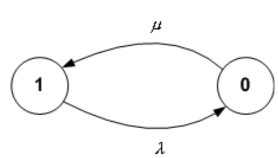
Hands-On Problem (5)

- Consider the single-component repairable system, find the state equations for computing steady-state probabilities and steady-state probabilities



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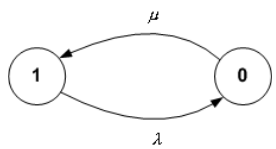
Asymptotic Solution (4)

- Steady-state equations ←→ Balance equations

  Rate entering = Rate leaving
- Example:
  - The balance equation is:



State	Rate leaving = rate entering
0:	$\mu P_0 = \lambda P_1$
1:	$\lambda P_1 = \mu P_0$

- Balance equations $\left.\begin{matrix} \\ \sum_{j=0}^{r} P_j(t) = 1 \end{matrix}\right\}$ ⇒ Steady-state probabilities P_i

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Solving State Equations (revisit)

- Unique solution to all the state probabilities $P_j(t)$ is obtained by solving

$$\begin{cases} \boldsymbol{A} \bullet \boldsymbol{P}(\boldsymbol{t}) = \dot{\boldsymbol{P}}(\boldsymbol{t}) \\ \sum_{j=0}^{r} P_j(t) = 1, \quad \text{the system must be in one of (r + 1) states} \\ P_i(0) = 1,\ P_k(0) = 0,\ \text{for } k = 0,\ldots,r \text{ and } k \neq i \quad \text{initial condition} \end{cases}$$

- Solutions

  √ Asymptotic (steady-state/long-run) solution

  – **Time-dependent solution**

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Time-Dependent Solution (1)

- State equations $\boldsymbol{A} \bullet \boldsymbol{P}(\boldsymbol{t}) = \dot{\boldsymbol{P}}(\boldsymbol{t})$
  - A set of linear, 1^{st}-order differential equations
  - The easiest and commonly used method is by Laplace transforms
- Laplace transforms
  - http://mathworld.wolfram.com/LaplaceTransform.html
  - http://www.vibrationdata.com/Laplace.htm
  - Any textbooks on mathematical analysis
  - Definition, some important properties, and some commonly-used transforms are written on board

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Time-Dependent Solution (2)

- Time-dependent solution using Laplace transform
  - $P_j^*(s)$: the Laplace transform of the state probability $P_j(t)$
  - According to the property: $L[f'(t)] = s \bullet L[f(t)] - f(0)$

$$L[\dot{P}_j(t)] = s \bullet L[P_j(t)] - P_j(0) = sP_j^*(s) - P_j(0)$$
$$\textbf{for } j = 0,1,\ldots,r$$

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Time-Dependent Solution (3)

- Thus, the Laplace transform of the state equations

$$L\left\{\begin{bmatrix} -\alpha_{00} & \alpha_{10} & \alpha_{20} & \dots\alpha_{r0} \\ \alpha_{01} & -\alpha_{11} & \alpha_{21} & \dots\alpha_{r1} \\ \alpha_{02} & \alpha_{12} & -\alpha_{22} & \dots\alpha_{r2} \\ : & : & : & : \\ \alpha_{0r} & \alpha_{1r} & \alpha_{2r} & \dots-\alpha_{rr} \end{bmatrix} \bullet \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ : \\ P_r(t) \end{bmatrix}\right\} = L\left\{\begin{bmatrix} \dot{P}_0(t) \\ \dot{P}_1(t) \\ \dot{P}_2(t) \\ : \\ \dot{P}_r(t) \end{bmatrix}\right\}$$

Based on Properties of Laplace transform:

$L[af(t)] = a \bullet L[f(t)];$ $\qquad L[f'(t)] = s \bullet L[f(t)] - f(0)$

$$\begin{bmatrix} -\alpha_{00} & \alpha_{10} & \alpha_{20} & \dots\alpha_{r0} \\ \alpha_{01} & -\alpha_{11} & \alpha_{21} & \dots\alpha_{r1} \\ \alpha_{02} & \alpha_{12} & -\alpha_{22} & \dots\alpha_{r2} \\ : & : & : & : \\ \alpha_{0r} & \alpha_{1r} & \alpha_{2r} & \dots-\alpha_{rr} \end{bmatrix} \bullet \begin{bmatrix} P_0{}^*(s) \\ P_1{}^*(s) \\ P_2{}^*(s) \\ : \\ P_r{}^*(s) \end{bmatrix} = \begin{bmatrix} sP_0{}^*(s) \\ sP_1{}^*(s) \\ sP_2{}^*(s) \\ : \\ sP_r{}^*(s) \end{bmatrix} - \begin{bmatrix} P_0(0) \\ P_1(0) \\ P_2(0) \\ : \\ P_r(0) \end{bmatrix} \quad \textbf{Eq. (LTSE)}$$

$$\sum_{i=0}^{r} P_i(t) = 1 \Rightarrow \sum_{i=0}^{r} P_i^*(s) = \frac{1}{s}$$

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Time-Dependent Solution (4)

- By introducing LT, the original state equations (a set of linear, **1st-order differential equations**) have been reduced to a set of **linear equations Eq. (LTSE)**
- Solving **Eq. (LTSE)** gives all **$P_j^*(s)$,** afterwards, the state probabilities **$P_j(t)$** can be determined from the inverse Laplace transforms!

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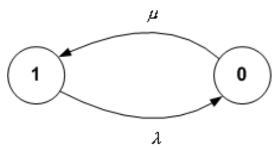
Hands-On Problem (6)

- Consider a repairable system composed of a single component, the failure rate of the component is λ, the repair rate is μ. The Markov chain model is



1: component is functioning
0: component is failed
Initial condition:
$P_1(0)=1$, $P_0(0)=0$

- Find the state probabilities $P_1(t)$ and $P_0(t)$

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Reliability Analysis Using Markov Chains

✓ Step 1: convert the fault tree model to a Markov chain

✓ Step 2: find the state equations of the Markov chain

✓ Step 3: find state probabilities by solving the state equations

- **Step 4: find the system reliability or unreliability**

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Step 4: Reliability Analysis Using Markov Chains

- System states S can be grouped into two sets
  - **O**: all states in which the system is operational
  - **F**: all states in which the system is failed **F = S - O**

$$\text{System reliability} = R_{sys} = \sum_{i \in O} P_i(t)$$

$$\text{System unreliability} = U_{sys} = \sum_{i \in F} P_i(t)$$

$$R_{sys} + U_{sys} = 1$$

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Hands-on Problems (7)

- Consider a parallel system of two independent and identical components with failure rate λ. When one of the components fails, it is repaired. The repair time is assumed to be exponentially distributed with repair rate μ. When both components have failed, the system is considered to have failed and no recovery is possible. Let the number of functioning components denote the state of the system. The state space is thus $\{0,1,2\}$. Assume the system to be in state 2 at time t=0.
  - Draw the state transition diagram for the system Markov chain.
  - Find the state equations for the time-dependent solution.
  - Find the state equations for the asymptotic solution.
  - Find the steady-state probabilities: P0, P1, P2
  - Find the Laplace transform of time-dependent state probabilities: P0*(s), P1*(s), P2* (s)

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Summary of Lecture #15

- A Markov process is a stochastic process with Markov property: probabilities of future states depend only on the current state and not on the history
- Any fault tree model (static or dynamic) with exponential component failure distribution can be solved as a Markov chain (with four steps)

Things to Do

- Homework
- ECE544 Project Report
  - Due **Wednesday, Nov. 30**
  - Please check out the Report Guidelines for requirements.