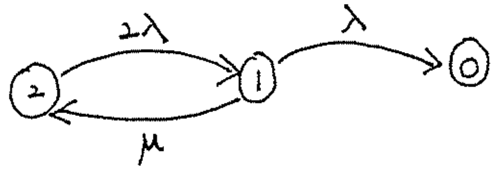
Problem #3

a.



b. The state equations are:

$$\begin{bmatrix} -\alpha_{00} & \alpha_{10} & \alpha_{20} \\ \alpha_{01} & -\alpha_{11} & \alpha_{21} \\ \alpha_{02} & \alpha_{12} & -\alpha_{22} \end{bmatrix} \begin{bmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix}$$

$$\begin{bmatrix} 0 & \lambda & 0 \\ 0 & -(\lambda+\mu) & 2\lambda \\ 0 & \mu & -2\lambda \end{bmatrix} \cdot \begin{bmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} \dot{p}_0(t) \\ \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix}$$

c. The steady-state equations are

$$\begin{bmatrix} 0 & \lambda & 0 \\ 0 & -(\lambda+\mu) & 2\lambda \\ 0 & \mu & -2\lambda \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

d. Find the steady-state probabilities P_0, P_1, P_2

According to the result of c, we have:

$$\begin{cases} P_0 * 0 + P_1 * [-(\lambda+\mu)] + P_2 * 2\lambda = 0 \\ P_0 * 0 + P_1 * \mu + P_2 * (-2\lambda) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -(\lambda+\mu) \cdot P_1 + 2\lambda \cdot P_2 = 0 & ① \\ \mu \cdot P_1 - 2\lambda \cdot P_2 = 0 & ② \end{cases}$$

plus $P_0 + P_1 + P_2 = 1$ ③

According to ②, $\mu P_1 = 2\lambda P_2 \Rightarrow P_2 = \frac{\mu}{2\lambda} P_1$.

According to ①: $-(\lambda+\mu) P_1 + 2\lambda \cdot \frac{\mu}{2\lambda} P_1 = 0 \Rightarrow P_1 = 0$

Thus, $P_2 = 0$

$P_0 = 1 - P_1 - P_2 = 1$

So, the steady-state probabilities are

$$\begin{cases} p_0 = 1 \\ p_1 = 0 \\ p_2 = 0 \end{cases}$$

Actually, the above results are obvious since state 0 is an absorbing state and reachable from the other states:

$$\lim_{t\to\infty} p_0(t) = p_0 = 1$$

e. Find the time-dependent state probabilities:

According to the result in (b), the transition rate matrix doesn't have full rank, we may remove one of the 3 equations w/o loosing any information about $p_0(t)$, $p_1(t)$, and $p_2(t)$. Here we remove the first of the 3 equations by removing the 1st row of the matrix and we get:

$$\begin{bmatrix} 0 & -(\lambda+\mu) & 2\lambda \\ 0 & \mu & -2\lambda \end{bmatrix} \begin{bmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix}$$

The matrix equations can be further reduced to: (since 1st column of matrix is zero)

$$\begin{bmatrix} -(\lambda+\mu) & 2\lambda \\ \mu & -2\lambda \end{bmatrix} \cdot \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix}$$

The Laplace transform of the above matrix equations are:

$$\begin{bmatrix} -(\lambda+\mu) & 2\lambda \\ \mu & -2\lambda \end{bmatrix} \cdot \begin{bmatrix} p_1^*(s) \\ p_2^*(s) \end{bmatrix} = \begin{bmatrix} s\,p_1^*(s) \\ s\,p_2^*(s) \end{bmatrix} - \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} s\,p_1^*(s) \\ s\,p_2^*(s) - 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -(\lambda+\mu)p_1^*(s) + 2\lambda p_2^*(s) = s\,p_1^*(s) \\ \mu\, p_1^*(s) - 2\lambda p_2^*(s) = s\,p_2^*(s) - 1 \end{cases}$$

solving equations for $p_1^*(s)$ and $p_2^*(s)$, we get

$$p_1^*(s) = \frac{2\lambda}{s^2+(3\lambda+\mu)\cdot s+2\lambda^2}$$

$$p_2^*(s) = \frac{\lambda+\mu+s}{s^2+(3\lambda+\mu)\cdot s+2\lambda^2}$$

Thus, $\quad p_0^*(s) = \frac{1}{s} - p_1^*(s) - p_2^*(s) \qquad (p_0(t) = 1 - p_1(t) - p_2(t))$

$$= \frac{2\lambda^2}{s\left[s^2+(3\lambda+\mu)\cdot s+2\lambda^2\right]}$$

HW solution Ends Here

Below is for your reference

The ~~=~~ state-probabilities are:

$$\begin{cases} p_0(t) = \mathcal{L}^{-1}[p_0^*(s)] \\ p_1(t) = \mathcal{L}^{-1}[p_1^*(s)] \\ p_2(t) = \mathcal{L}^{-1}[p_2^*(s)] \end{cases}$$

To find the inverse Laplace transforms, we write

$s^2+(3\lambda+\mu)\cdot s+2\lambda^2 = (s-k_1)(s-k_2)$, where

$$k_1 = \frac{-(3\lambda+\mu)+\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}$$

$$k_2 = \frac{-(3\lambda+\mu)-\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}$$

The expression for $p_1^*(s)$ can be rearranged so that

$$p_1^*(s) = \frac{2\lambda}{k_1-k_2}\left(\frac{1}{s-k_1} - \frac{1}{s-k_2}\right)$$

By inverting this transform, we get

$$p_1(t) = \frac{2\lambda}{k_1-k_2}\left(e^{k_1t} - e^{k_2t}\right) \qquad \left[(*)\ \mathcal{L}[e^{\alpha t}] = \frac{1}{s-\alpha}\right]$$

Similarly, we can rewrite the expression for $p_0^*(s)$ as:

$$p_0^*(s) = \frac{2\lambda^2}{s} \times \frac{1}{k_1-k_2}\left[\frac{1}{s-k_1} - \frac{1}{s-k_2}\right]$$

$$= \frac{2\lambda^2}{k_1-k_2}\left[\frac{1}{s(s-k_1)} - \frac{1}{s(s-k_2)}\right]$$

$$= \frac{2\lambda^2}{k_1-k_2}\left[\left(\frac{1}{s-k_1} - \frac{1}{s}\right)\cdot\frac{1}{k_1} - \left(\frac{1}{s-k_2} - \frac{1}{s}\right)\cdot\frac{1}{k_2}\right]$$

$$= \frac{2\lambda^2}{k_1(k_1-k_2)}\left[\frac{1}{s-k_1} - \frac{1}{s}\right] - \frac{2\lambda^2}{(k_1-k_2)k_2}\left(\frac{1}{s-k_2} - \frac{1}{s}\right)$$

$$\therefore\ p_0(t) = \frac{2\lambda^2}{k_1(k_1-k_2)}\left[e^{k_1 t} - 1\right] - \frac{2\lambda^2}{(k_1-k_2)k_2}\left[e^{k_2 t} - 1\right]$$

$$= \frac{2\lambda^2}{k_1 k_2(k_1-k_2)}\left[k_2 e^{k_1 t} - k_2 - k_1 e^{k_2 t} + k_1\right]$$

Finally,

$$p_2(t) = 1 - p_1(t) - p_0(t)$$