☆ Defn: Let $f(t)$ be a function defined on the interval $(0, \infty)$. The Laplace transform of the function $f(t)$ is defined by

$$\triangle \quad \underline{f^*(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt}$$

note: ① when $f(t)$ is the p.d.f. of a non-negative r.v. T, the LT of $f(t)$ is seen to be equal to the expected value of the r.v. e^{-sT}

$$E[e^{-sT}] = \int_0^\infty e^{-st} f(t)\,dt = f^*(s)$$

$$E[T] = \int_0^\infty t f(t)\, dt$$

② The function $f(t)$ is called the inverse Laplace transform of $f^*(s)$ and is written

$$\triangle \quad \underline{f(t) = \mathcal{L}^{-1}[f^*(s)]}$$

☆ Some commonly-used Laplace transform:

$f(t)$, $t \geq 0$	$f^*(s) = \mathcal{L}[f(t)]$	
✓ 1	$1/s$	
✓ t	$1/s^2$	
✓ t^2	$2!/s^3$	
✓ t^n	$n!/s^{n+1}$	for $n = 0, 1, 2, \cdots$
✓ $e^{\alpha t}$	$\frac{1}{s-\alpha}$	α is a constant, $s > \alpha$
$e^{\alpha t} \cdot t^n$	$\frac{n!}{(s-\alpha)^{n+1}}$	$s > \alpha$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	
$e^{\alpha t} \sin(\beta t)$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$	$s > \alpha$
$e^{\alpha t} \cos(\beta t)$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$	$s > \alpha$

\# Some important properties of the Laplace transform:

1. $\mathcal{L}[f_1(t) + f_2(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)]$

2. $\mathcal{L}[\alpha f(t)] = \alpha \mathcal{L}[f(t)]$ α is a constant

3. $\mathcal{L}[f(t-\alpha)] = e^{-\alpha s}\mathcal{L}[f(t)]$

4. $\mathcal{L}[e^{\alpha t} f(t)] = f^*(s-\alpha)$

5. $\mathcal{L}[f'(t)] = s \cdot \mathcal{L}[f(t)] - f(0) = s f^*(s) - f(0)$

6 $\mathcal{L}[\int_0^t f(u)du] = \mathcal{L}[f(t)]/s$

7 $\mathcal{L}[\underbrace{\int_0^\infty f_1(t-u) f_2(u) du}_{f_1(t) * f_2(t) \text{ convolution}}] = \mathcal{L}[f_1(t)] \cdot \mathcal{L}[f_2(t)]$

8. $\lim_{s\to\infty} s f^*(s) = \lim_{t\to 0} f(t)$

9 $\lim_{s\to 0} s f^*(s) = \lim_{t\to\infty} f(t)$